

## 3. Photons and phonons

Kai Hock

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University of Liverpool

### 3.1 Phonons

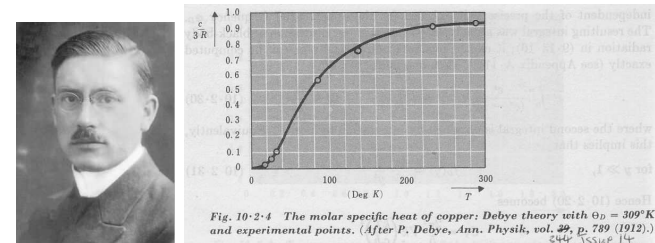
### 3.1 Phonons

### 3.2 Photons

### 3.3 Exercises

### Specific heat capacities of solids

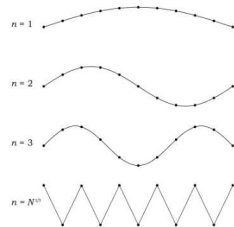
Previously, we have looked at Einstein's model of the 3-D simple harmonic oscillator, in which each atom in a solid is in its own potential well. The predicted heat capacity was accurate at high temperature, but fails at low temperature.



In 1912, Peter Debye, a Dutch physicist working in Germany, produced a theory which turns out to be very accurate both hot and cold.

In Debye's model, we think of the atoms in a solid as connected by springs. They vibrate in 3D in a complicated way.

The problem can be simplified by treating the vibration as a superposition of waves of different frequencies.



If we suppose that the atoms are fixed at the edge of the solid, the frequency would be quantised. A quantum of this vibration energy is called a phonon.

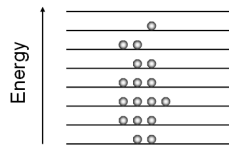
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### Bose-Einstein Statistics

The method is similar to what we have used for the ideal gas.

There, we have made the assumption that there are many more energy levels than there are particles. This has meant that it is unlikely for two gas atoms to occupy the same state.

We do not make this assumption now, since we would be particularly interested in the low temperature behaviour.



Since the exclusion principle does not apply to phonons, more than one phonons can occupy the same state.

We can count the number of states for the phonons, like we did for atoms in an ideal gas, or electrons in a metal.

Unlike electrons, however, the exclusion principle does not apply to phonons.

So each energy state can be occupied by any number of phonons.

This means that we need yet another way to find the macrostate. This was worked out by Bose and Einstein in the 1920s.

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### Bose-Einstein Statistics

Suppose, in the energy bundle  $i$ , that there are  $g_i$  levels and  $n_i$  phonons. The phonons are now completely free to arrange themselves among the energy levels.

This means that we have  $n_i + g_i$  objects altogether in the bundle. There are  $n_i$  of one type, and  $g_i$  of the other type. The number of possible arrangements would be

$$\Omega_i = \frac{(n_i + g_i)!}{n_i!g_i!}$$

The total number of arrangements for all bundles would be obtained by multiplication of the number for every bundle:

$$\Omega = \prod_i \frac{(n_i + g_i)!}{n_i!g_i!}.$$

Then we need to maximise  $\ln \Omega$  using the Lagrange multiplier method.

Using the same constraints as before on the number of particles  $N$  and the total energy  $U$ , the method would give the following answer:

$$\frac{n_i}{g_i} = \frac{1}{\exp(-\lambda_1 - \lambda_2 \varepsilon_i) - 1}.$$

where  $\lambda_1$  and  $\lambda_2$  are the Lagrange multipliers.

This is called the Bose-Einstein distribution function. It looks almost the same as the Fermi-Dirac distribution function, except that it has -1 in the denominator instead of +1.

When we apply this to phonon, however, we need to make some changes.

### Phonon occupation number

Maximising Lagrange function, we have obtained

$$\frac{n_i}{g_i} = \frac{1}{\exp(\varepsilon_i/k_B T) - 1}.$$

Following the same notations that we have used for ideal gas and electrons, we rewrite:

$n_i$  as  $n(\varepsilon)d\varepsilon$  and

$g_i$  as  $g(\varepsilon)d\varepsilon$ .

This gives

$$n(\varepsilon)d\varepsilon = \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T) - 1}$$

Next, we need a formula for the density of states  $g(\varepsilon)$ .

Note that although we have previously used the same symbol  $g(\varepsilon)$  for the particle in a 3-D box, the phonon density of states would have a different formula.

Phonons are not real particles like atoms or electrons. The number of phonons is related to the vibration energy. The number would increase when the temperature increases. At 0 Kelvin, there is no vibration, so there would be no phonons.

Because of this, there would be no constraint on the number  $N$ . We really only have one constraint - that of constant energy  $U$ .

The Lagrange function is  $\ln \Omega + \lambda U$ .  $N$  does not appear.

Maximising this function, we get

$$\frac{n_i}{g_i} = \frac{1}{\exp(-\lambda \varepsilon_i) - 1}.$$

With the help of entropy as before, we would find

$$\lambda = -\frac{1}{k_B T}.$$

### density of states for phonons

The vibrations in a solid is described by a wave equation. The solution for the displacement of the atoms  $\xi$ , can be given by the same function

$$\xi = \sin k_x x$$

in each of the  $x$ ,  $y$  and  $z$  directions. Assuming that the displacements are zero at the edge of the solid, the wavevectors are discretised in the same way:

$$k_x = \frac{n_x \pi}{a}.$$

So the density of states can also be given by the same formula:

$$g(k) = \frac{V k^2}{2\pi^2}.$$

However, two changes are needed from the way we have used it for the ideal gas and the electrons.

The first change is in the change of variable to energy  $\epsilon$ .

For atoms or electrons, the energy is

$$\epsilon = \frac{\hbar^2 k^2}{2m}.$$

For vibrations, the energy is given by the same formula as for photons:

$$\epsilon = \hbar\omega,$$

where  $\omega$  is the angular frequency  $2\pi f$  and  $f$  is the frequency.  $\omega$  is in turn related to  $k$  by

$$v = \frac{\omega}{k},$$

where  $v$  is the velocity.

### About polarisation

The second change is related to the polarisation. Recall that for electrons, we have to multiply by 2 because of the 2 spin states.

For vibration, a state is a particular frequency, for a particular set of wavevectors  $(k_x, k_y, k_z)$ .

Each state corresponds to a wave, and a wave in a solid can have 2 transverse polarisations, and 1 longitudinal polarisation.

So we have to multiply by 3 to get the correct density of state:

$$g(\omega)d\omega = 3 \times \frac{V\omega^2 d\omega}{2\pi^2 v^3}$$

The variable of angular frequency  $\omega$  is often used instead of energy  $\epsilon$ . For the change of variable, we then use

$$\omega = vk.$$

We can now do the change of variable using

$$g(\omega)d\omega = g(k)dk.$$

Substituting

$$g(k) = \frac{Vk^2}{2\pi^2},$$

we get the density of states in  $\omega$ :

$$g(\omega) = \frac{V\omega^2}{2\pi^2 v^3}.$$

### The Debye Frequency

There is one more practical point to note. In a solid, there is a limit to the highest phonon frequency.

This is related to the fact that the wavelength cannot possibly be shorter than the distance between atoms.

Let this highest frequency be  $\omega_D$ .

In a solid with  $N$  identical atoms, there are  $3N$  energy states, or normal modes. This is when we count every possible frequency, and every possible "direction of vibration" for each frequency.

This can be proven mathematically, but we shall skip that. We can find  $\omega_D$  by integrating the density of states:

$$\int_0^{\omega_D} g(\omega)d\omega = \int_0^{\omega_D} \frac{3V\omega^2 d\omega}{2\pi^2 v^3} = 3N$$

This equation

$$\int_0^{\omega_D} g(\omega) d\omega = \int_0^{\omega_D} \frac{3V\omega^2 d\omega}{2\pi^2 v^3} = 3N$$

can then be solved to find the Debye frequency. Integrating gives

$$\frac{V\omega_D^3}{2\pi^2 v^3} = 3N$$

Solving for  $\omega_D$  gives the Debye frequency:

$$\omega_D = \left( \frac{6N\pi^2 v^3}{V} \right)^{1/3}$$

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Debye's theory for Heat capacity of solid

The phonon distribution is given by

$$n(\omega) d\omega = \frac{g(\omega) d\omega}{\exp(\hbar\omega/k_B T) - 1}.$$

We can multiply by  $\hbar\omega$  to get the energy at this frequency interval:

$$\hbar\omega n(\omega) d\omega = \frac{g(\omega) d\omega}{\exp(\hbar\omega/k_B T) - 1}.$$

The total energy is then

$$U = \int_0^{\omega_D} \hbar\omega n(\omega) d\omega = \int_0^{\omega_D} \frac{\hbar\omega g(\omega) d\omega}{\exp(\hbar\omega/k_B T) - 1}.$$

We have found earlier that the density of states is:

$$g(\omega) = 3 \times \frac{V\omega^2}{2\pi^2 v^3}.$$

Substituting into the integral for the total energy above ...

As we have shown earlier, the distribution of phonons is given by

$$n(\varepsilon) d\varepsilon = \frac{g(\varepsilon) d\varepsilon}{\exp(\varepsilon/k_B T) - 1}$$

We shall express this in terms of frequency  $\omega$  as well. The phonon energy  $\varepsilon$  is given by

$$\varepsilon = \hbar\omega.$$

This comes from assuming that the atoms vibrate in a harmonic potential, so that the energy is quantised in steps of  $\hbar\omega$ .

In terms of the frequency  $\omega$ , the phonon distribution is then given by

$$n(\omega) d\omega = \frac{g(\omega) d\omega}{\exp(\hbar\omega/k_B T) - 1}.$$

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Debye's theory for Heat capacity of solid

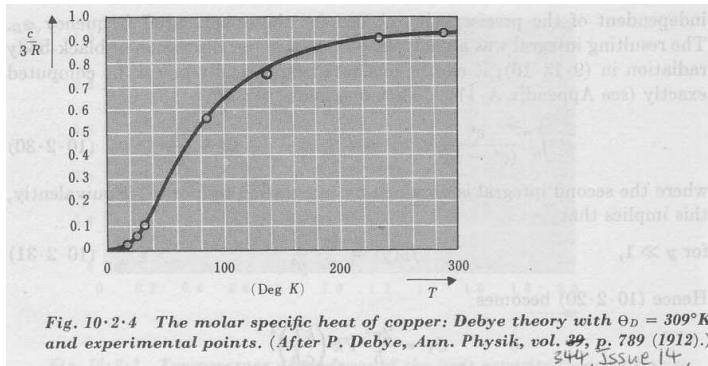
We get:

$$U = \frac{3V\hbar}{2\pi^2 v^3} \int_0^{\omega_D} \frac{\omega^3 d\omega}{\exp(\hbar\omega/k_B T) - 1}$$

At low temperature, it can be shown that this is proportional to  $T^4$ . This would imply that the heat capacity  $C$  is proportional to  $T^3$ . (This has been mentioned in the topic on the specific heat of electrons in metal.)

At high temperatures, the heat capacity tends to  $3Nk_B$ .

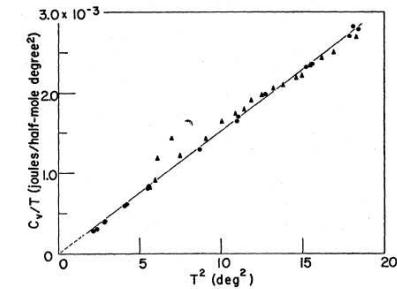
In 1912, Debye measured the high temperature behaviour of copper.



He showed that the heat capacity did tend to  $3Nk_B$ , as he had predicted.

### 3.2 Photons

In 1953, at Purdue University, Keesom and Pearlman measured the low temperature behaviour of potassium chloride.



P. H. Keesom and N. Pearlman, *Physical Review*, vol. 91 (1953), pp. 1354-1355

As predicted by Debye, the heat capacity was indeed proportional to  $T^3$ .

When an object gets very hot, it can give out light.

The temperature of a volcano lava flow can be estimated by observing its color. The result agrees well with the measured temperatures of lava flows at about 1,000 to 1,200 deg C.



[http://en.wikipedia.org/wiki/Black\\_body](http://en.wikipedia.org/wiki/Black_body)

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## An ideal black body

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The amount of radiation emitted depends on the nature of the object - its colour, whether it is smooth or rough, etc.

An ideal black body is one that absorbs all the radiation that falls on it.

No real material can do this. Soot is about the best, absorbing all but 3%.

In 1859 Gustav Kirchhoff had a good idea: "a small hole in the side of a large box is an excellent absorber, since any radiation that goes through the hole bounces around inside, a lot getting absorbed on each bounce, and has little chance of ever getting out again."

[http://galileo.phys.virginia.edu/classes/252/black\\_body\\_radiation.html](http://galileo.phys.virginia.edu/classes/252/black_body_radiation.html)

The energy of the absorbed radiation reaches equilibrium among different frequencies in the cavity. This results in a characteristic spectrum that would be emitted again through the hole.

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## Photons in a box

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We can derive a formula for the black body radiation by considering the energies of a free photon gas in a box (following Kirchhoff's idea).

Photons arise from oscillations in electromagnetic fields, which is also governed by a wave equation.

The same ideas for phonons can be used, leading to the same density of states:

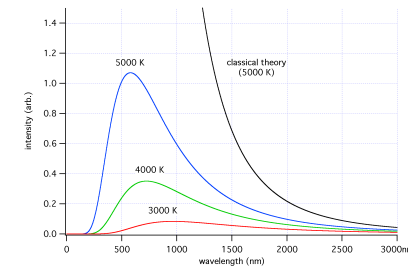
$$g(\omega) = \frac{V\omega^2}{2\pi^2c^3},$$

where the speed  $v$  is now the speed of light  $c$ . This is the formula before considering polarisations.

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## Black Body Spectrum

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[http://en.wikipedia.org/wiki/Black\\_body](http://en.wikipedia.org/wiki/Black_body)

As the temperature decreases, the peak of the black-body radiation curve moves to lower intensities and longer wavelengths. The black-body radiation graph is also compared with the classical model of Rayleigh and Jeans.

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## Photons in a box

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Like phonons, the photon number is not fixed. We get few photons when the box is cold, and more photons when it is hot.

Unlike phonons, there is no upper limit to the frequency (no Debye frequency). The wavelength in a solid cannot be shorter than the distance between atoms. The electromagnetic wave in a box has no such limit.

Finally, phonons in a solid can have 3 polarisations: 2 transverse and 1 longitudinal (like sound). Photons can only have 2: both transverse (e.g. light).

Recall that for phonons, the density of states is give by:

$$g(\omega)d\omega = 3 \times \frac{V\omega^2 d\omega}{2\pi^2 v^3}$$

where the factor of 3 comes from the 3 polarisations of a phonon, 2 transverse and 1 longitudinal.

Since photons only have 2 polarisations, both transverse, the 3 should be replaced by a 2:

$$g(\omega)d\omega = 2 \times \frac{V\omega^2 d\omega}{2\pi^2 c^3}$$

where the sound speed  $v$  is replaced by light speed  $c$ .

The energy of a photon is  $\hbar\omega$ . The energy in the frequency interval  $d\omega$  is then

$$\hbar\omega n(\omega)d\omega = 2 \times \hbar\omega \times \frac{V\omega^2 d\omega}{2\pi^2 c^3} \times \frac{d\omega}{\exp(\hbar\omega/k_B T) - 1}$$

The energy density is given by:

$$u(\omega) = \hbar\omega n(\omega) = \frac{V\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/k_B T) - 1}.$$

This is essentially Planck's law for black body radiation.

Integrating gives the total energy:

$$U = \frac{\pi^2 V k_B^4}{15 \hbar^3 c^3} T^4$$

It can be shown that this leads to Stefan's law of radiation:

$$\eta = \sigma T^4$$

for radiation emitted by an object at a temperature  $T$ .

Like phonons, each energy state can be occupied by any number of photons.

So it obeys Bose-Einstein statistics.

The number density is therefore given by the same formula:

$$n(\epsilon)d\epsilon = \frac{g(\epsilon)d\epsilon}{\exp(\epsilon/k_B T) - 1}$$

So the number of photons in a given frequency interval is

$$n(\omega)d\omega = 2 \times \frac{V\omega^2 d\omega}{2\pi^2 c^3} \times \frac{d\omega}{\exp(\hbar\omega/k_B T) - 1}$$

Berlin, 1899. In the Imperial Institute of Physics and Technology, the scientists were just trying to help the German lighting and heating industries.



**Otto  
Lummer**  
1860-1925

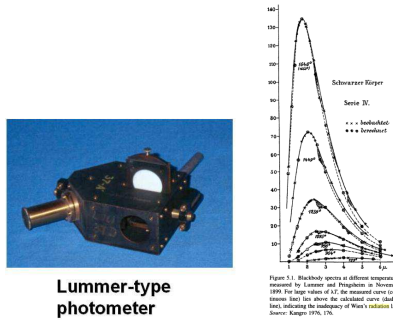


**Ernst  
Pringsheim**  
1859-1917

Lummer and Pringsheim measured the radiation spectrum from a blackbody very accurately.



The results did not look at all like what the physicists thought it should.



Then Max Planck came along and derived his formula for black body radiation that agreed very well with these measurements. And the rest is history.

What we have learnt so far.

electrons

$$g(\varepsilon) = 2 \times \frac{4m\pi V}{h^3} (2m\varepsilon)^{1/2}$$

classical ideal gas

$$g(\varepsilon) = \frac{4m\pi V}{h^3} (2m\varepsilon)^{1/2}$$

3. At low temperature, the specific heat capacity for the Debye model tends to  $T^3$ . At high temperature, it tends to  $3Nk_B$ .

4. The energy distribution in a photon gas is given by:

$$u(\omega) = \hbar\omega n(\omega) = \frac{V\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/k_B T) - 1}.$$

This also describes the black body radiation spectrum.

What we have learnt so far.

1. The Bose-Einstein distribution function is given by

$$f(\varepsilon) = \frac{1}{\exp[(\varepsilon - \mu)/k_B T] - 1}.$$

2. For phonons and photons, the following form is used:

$$f(\varepsilon) = \frac{1}{\exp(\varepsilon/k_B T) - 1}.$$

3. The density of states can take different forms:

phonons

$$g(\omega)d\omega = 3 \times \frac{V\omega^2 d\omega}{2\pi^2 v^3}$$

photons

$$g(\omega)d\omega = 2 \times \frac{V\omega^2 d\omega}{2\pi^2 v^3}$$

bosonic atoms

$$g(\varepsilon) = \frac{4m\pi V}{h^3} (2m\varepsilon)^{1/2}$$

### 3.3 Exercises

## Exercise 1

Sketch the density of states as a function of frequency for 1, 2 and 3-D solids in the Debye approximation. Consider a 2-D solid. Show according to the Debye theory that the heat capacity varies as  $T^2$  at low temperature. In graphite it is found that  $C_V \propto T^{2.4}$  at low temperature. Explain this behaviour.

## density of states for phonons

Now we repeat the steps for 2-D. First, we obtain the quantised energy states (no  $n_z$ ):

$$\frac{\omega^2 a^2}{v^2 \pi^2} = n_x^2 + n_y^2.$$

Then we compare with the equation for a circle (not sphere)

$$R^2 = n_x^2 + n_y^2$$

For positive integers  $(n_x, n_y)$ , the total number of states with frequency less than  $\omega$  is given by the quadrant of the circle (not octant):

$$G(\omega) = \frac{1}{4}(\pi R^2) = \frac{\pi}{4} \left( \frac{\omega a}{v\pi} \right)^2$$

Differentiating, we get the density of states:

$$g(\omega) = 2 \times \frac{A\omega}{2\pi v^2}$$

where  $A$  is the area  $a^2$  (not volume). There is an extra factor of 2 because in 2-D, there can be 2 modes for each phonon frequency: 1 transverse and 1 longitudinal.

Recall the steps leading to the density of states of a phonon in a 3-D solid. First, we obtain the quantised energy states:

$$\frac{\omega^2 a^2}{v^2 \pi^2} = n_x^2 + n_y^2 + n_z^2.$$

Then we compare with the equation for a sphere

$$R^2 = n_x^2 + n_y^2 + n_z^2$$

For positive integers  $(n_x, n_y, n_z)$ , the total number of states with frequency less than  $\omega$  is given by the octant of the sphere:

$$G(\omega) = \frac{1}{8} \left( \frac{4\pi}{3} R^3 \right) = \frac{\pi}{6} \left( \frac{\omega a}{v\pi} \right)^3$$

Differentiating, we get the density of states:

$$g(\omega) = 3 \times \frac{V\omega^2}{2\pi^2 v^3}$$

where  $V$  is the volume  $a^3$ . There is an extra factor of 3 because in 3-D, there can be 3 modes for each phonon frequency: 2 transverse and 1 longitudinal.

## Exercises

Finally we repeat the steps for 1-D. First, we obtain the quantised energy states (no  $n_y$ ):

$$\frac{\omega^2 a^2}{v^2 \pi^2} = n_x^2.$$

Then we compare with the equation for a line (not circle)

$$R^2 = n_x^2$$

For positive integers  $n_x$ , the total number of states with frequency less than  $\omega$  is given by half of the line (not quadrant):

$$G(\omega) = \frac{1}{2}(2R) = \frac{\omega a}{v\pi}$$

Differentiating, we get the density of states:

$$g(\omega) = 1 \times \frac{L}{\pi v}$$

where  $L$  is the length  $a$  (not area). There is no additional factor because in 1-D, there can be only be 1 mode for each phonon frequency: the longitudinal.

Summarising, we have obtained the following results for the density of states:

$$\begin{aligned} \text{1-D:} & \quad g(\omega) = \text{constant} \\ \text{2-D:} & \quad g(\omega) \propto \omega \\ \text{3-D:} & \quad g(\omega) \propto \omega^2 \end{aligned}$$

The internal energy is given by:

$$U = \int_0^{\omega_D} \frac{\hbar \omega g(\omega) d\omega}{\exp(\hbar \omega / k_B T) - 1}.$$

In 2-D,

$$g(\omega) \propto \omega.$$

So

$$U \propto \int_0^{\omega_D} \frac{\omega^2 d\omega}{\exp(\hbar \omega / k_B T) - 1}.$$

Since the definite integral here

$$U \propto T^3 \int_0^{\infty} \frac{x^2 dx}{e^x - 1}.$$

is a constant, we have

$$U \propto T^3.$$

Differentiating, the heat capacity is

$$C_V \propto T^2.$$

Graphite has a layered structure. Each layer consists of carbon atoms strongly bonded, with weak forces between layers. Vibration (like sound) would propagate much faster along the layer than across the layers. So it can be thought of as intermediate between 2-D and 3-D.

This explains why the heat capacity of graphite is

$$C_V \propto T^{2.4}.$$

We have obtained

$$U \propto \int_0^{\omega_D} \frac{\omega^2 d\omega}{\exp(\hbar \omega / k_B T) - 1}.$$

Let

$$x = \frac{\hbar \omega}{k_B T}.$$

At low temperature,  $T \rightarrow 0$  and  $x \rightarrow \infty$ . The integral for  $U$  can be written as

$$U \propto T^3 \int_0^{\infty} \frac{x^2 dx}{e^x - 1}.$$

The factor of  $T^3$  appears because of the factor of  $T$  in the substitution for  $\omega$ :

$$\omega = \frac{k_B T}{\hbar} x.$$

## Exercise 2

According to Debye's theory, the total energy of phonons in a solid is given by

$$U = \frac{3V\hbar}{2\pi^2 v^3} \int_0^{\omega_D} \frac{\omega^3 d\omega}{\exp(\hbar \omega / k_B T) - 1}.$$

Show that at low temperatures, i.e.  $T \ll \hbar \omega_D / k_B$ , Debye's theory predicts that the heat capacity of a solid is proportional to  $T^3$ .

[ You are given that

$$\int_0^{\infty} \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}.$$

]

To use the formula given, let the argument of the exponential function be  $x$ :

$$x = \frac{\hbar\omega}{k_B T}.$$

Make frequency the subject:

$$\omega = \frac{k_B T}{\hbar} x.$$

Then substitute into the integral for the energy:

$$U = \frac{3V\hbar}{2\pi^2 v^3} \left(\frac{k_B T}{\hbar}\right)^4 \int_0^{x_D} \frac{x^3 dx}{e^x - 1}$$

where the new limit is related to the old one by the substitution formula:

$$x_D = \frac{\hbar\omega_D}{k_B T}.$$

We have found the total energy.

$$U = \frac{V\pi^2 k_B^4 T^4}{10\hbar^3 v^3}$$

Note that this is proportional to  $T^4$ . To get the heat capacity, we differentiate with respect to temperature  $T$ :

$$C = \frac{2V\pi^2 k_B^4 T^3}{5\hbar^3 v^3}$$

which is proportional to  $T^3$ .

This answers the question.

The condition given is  $T \ll \hbar\omega_D/k_B$ . Rearranging, we have

$$1 \ll \frac{\hbar\omega_D}{k_B T}.$$

The right hand side is just  $x_D$ , so

$$1 \ll x_D.$$

This means  $x_D$  is large. We may approximate the integral by replacing the limit by infinity:

$$U \approx \frac{3V\hbar}{2\pi^2 v^3} \left(\frac{k_B T}{\hbar}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}.$$

Then using the formula given,

$$U = \frac{3V\hbar}{2\pi^2 v^3} \left(\frac{k_B T}{\hbar}\right)^4 \frac{\pi^4}{15}.$$

Some remarks:

Going back to our approximation, how large must  $x_D$  be for the approximation

$$\int_0^{x_D} \frac{x^3 dx}{e^x - 1} \approx \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

to be valid? We can get some idea if we carry out the integration numerically:

$x_D$	$\int \frac{x^3 dx}{e^x - 1}$	error
1	0.225	96%
10	6.432	1%
100	6.494	$10^{-18}$
$\infty$	$\pi^4/15$	0

As long as  $x_D$  is more than 10, the approximation would be better than 1%.

Recall that  $x_D$  is related to the Debye frequency by

$$x_D = \frac{\hbar\omega_D}{k_B T},$$

This means that for 1% accuracy, we should have

$$\frac{\hbar\omega_D}{k_B T} > 10.$$

Rearranging, we get

$$T < \frac{\theta_D}{10}$$

where

$$\theta_D = \frac{\hbar\omega_D}{k_B}.$$

$\theta_D$  is called the Debye temperature, and can be used as a reference when we want to tell whether a temperature is high or low. Note that in the numerator is  $\hbar\omega_D$ , which is the highest possible phonon energy.

We must rewrite the distribution in terms of wavelength  $\lambda$ :

$$\omega = 2\pi f = 2\pi \frac{c}{\lambda}$$

Differentiating,

$$d\omega = -\frac{2\pi c}{\lambda^2} d\lambda.$$

Substituting into the energy distribution:

$$u(\lambda)d\lambda = \frac{Vh}{\pi^2 c^2 \lambda^5} \frac{d\lambda}{\exp(hc/\lambda k_B T) - 1}.$$

To find the maximum, we can ignore the constants in  $u(\omega)$  and just focus on the part that depends on  $\omega$ :

$$\frac{1}{\lambda^5} \frac{1}{\exp(hc/\lambda k_B T) - 1}.$$

To find the maximum, differentiate this ...

### Exercise 3

The energy distribution of a gas of photons in a box given by:

$$u(\omega)d\omega = \frac{Vh}{\pi^2 c^3} d\omega \frac{\omega^3}{\exp(\hbar\omega/k_B T) - 1}.$$

Wien's displacement law for black body radiation is given by

$$\lambda_{\max} = \frac{b}{T},$$

where  $\lambda_{\max}$  is the wavelength of the peak of the radiation intensity, and  $b$  is  $2.90 \times 10^{-3}$  m.K.

Derive Wien's displacement law from the energy distribution.

and set to zero:

$$-\frac{5}{\lambda^6} \frac{1}{\exp(hc/\lambda k_B T) - 1} - \frac{1}{\lambda^5} \frac{(-hc/\lambda^2 k_B T) \exp(hc/\lambda k_B T)}{[\exp(hc/\lambda k_B T) - 1]^2} = 0.$$

Multiplying by  $\lambda^6 [\exp(hc/\lambda k_B T) - 1]^2$ ,

$$-5[\exp(hc/\lambda k_B T) - 1] + \frac{hc}{\lambda k_B T} \exp(hc/\lambda k_B T) = 0.$$

We can simplify this by defining

$$x = \frac{hc}{\lambda k_B T}.$$

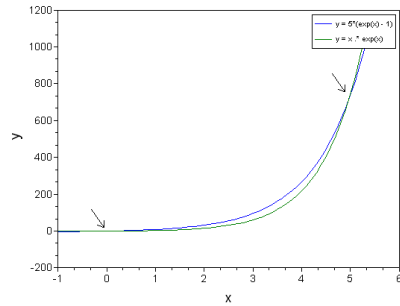
Substituting, we get

$$-5(e^x - 1) + xe^x = 0.$$

If we plot each side of the equation,

$$5(e^x - 1) = xe^x.$$

we see that there are 2 solutions.



We can solve this numerically. The answers are

$$x_1 = 0 \quad \text{and} \quad x_2 = 4.965.$$

The nontrivial solution is

$$x_2 = 4.965.$$

Substituting this into the definition for  $x$ ,

$$x = \frac{h}{\lambda k_B T}.$$

or making  $\lambda$  the subject, we get

$$\lambda = \frac{hc}{x_2 k_B T}.$$

Comparing this with the displacement law,

$$\lambda_{\max} = \frac{b}{T},$$

we see that

$$b = \frac{hc}{x_2 k_B}.$$

Substituting the constants and calculating, we get the Wien's displacement constant  $2.90 \times 10^{-3} \text{ m.K.}$